Arrow's Impossibility Theorem
Lecture 12
Section 1.6

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Outline

The Majority Criterion

Definition (Majority Criterion)

The Majority Criterion says that if a candidate has a majority of first-place votes, then that candidate should be the winner. (There may or may not be a majority candidate.)

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The Majority Criterion says that if a candidate has a majority of first-place votes, then that candidate should be the winner. (There may or may not be a majority candidate.)

- If no candidate has a majority, then the Majority Criterion cannot be violated.
- The Borda count method and Coombs' method may violate the Majority Criterion.

The Majority Criterion

Example (The Majority Criterion – Borda Count Method)

No. of Votes	8	6	5	3
1st	A	A	B	B
2nd	B	B	C	C
3rd	C	D	D	A
4th	D	C	A	D

- Does any candidate have a majority? If so, "should" win?

The Majority Criterion

Example (The Majority Criterion – Borda Count Method)

No. of Votes	8	6	5	3
1st	A	A	B	B
2nd	B	B	C	C
3rd	C	D	D	A
4th	D	C	A	D

- Does any candidate have a majority? If so, "should" win?
- Who wins by the Borda Count Method?

The Majority Criterion

Example (The Majority Criterion – Coomb's Method)

No. of Votes	8	6	5	2
1st	D	A	A	B
2nd	B	B	C	C
3rd	C	D	D	A
4th	A	C	B	D

- Does any candidate have a majority? If so, who "should" win?

Example (The Majority Criterion – Coomb's Method)

No. of Votes	8	6	5	2
1st	D	A	A	B
2nd	B	B	C	C
3rd	C	D	D	A
4th	A	C	B	D

- Does any candidate have a majority? If so, who "should" win?
- Who wins by Coomb's Method?

The Majority Criterion

- The Borda count method and Coombs' method may violate the Majority Criterion.
- The other methods (I'm pretty sure) do not violate it.

Outline

The Condorcet Criterion

Definition (Condorcet Winner)

The Condorcet winner is a candidate who beat *every other candidate* in pairwise comparisons. (There may or may not be a Condorcet winner.)

Definition (Condorcet Criterion)

The Condorcet Criterion says that if there is a Condorcet winner, then that candidate should be the winner (by whatever method used).

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The Condorcet Criterion says that if there is a Condorcet winner, then that candidate should be the winner (by whatever method used).

- If there is no Condorcet winner, then the Condorcet Criterion cannot be violated.

The Condorcet Criterion

Example (The Condorcet Criterion – Borda Count Method)

No. of Votes	8	6	5	3
1st	A	A	B	B
2nd	B	B	C	C
3rd	C	D	D	A
3rd	D	C	A	D

- Is there a Condorcet winner? If so, who "should" win?

Example (The Condorcet Criterion – Borda Count Method)

No. of Votes	8	6	5	3
1st	A	A	B	B
2nd	B	B	C	C
3rd	C	D	D	A
3rd	D	C	A	D

- Is there a Condorcet winner? If so, who "should" win?
- Who wins by the Borda Count Method?

- The plurality method, Borda count method, plurality-with-elimination method, and Coombs' method may violate the Condorcet Criterion.
- The method of pairwise comparisons does not violate it.

Outline

The Monotonicity Criterion

Definition (Monotonicity Criterion)

The Monotonicity Criterion says that if candidate X is the winner, then X would still be the winner if a voter had placed X higher in his ranking.

The Monotonicity Criterion

Example (The Monotonicity Criterion – Plurality-with-Elimination Method)

	10	8	7	4
1st	B	A	C	C
2nd	C	B	A	B
3rd	A	C	B	A

- Who is the winner by the Plurality-with-Elimination Method? If so, who "should" win?

The Monotonicity Criterion

Example (The Monotonicity Criterion – Plurality-with-Elimination Method)

	10	8	7	4
1st	B	A	C	C
2nd	C	B	A	B
3rd	A	C	B	A

- Who is the winner by the Plurality-with-Elimination Method? If so, who "should" win?
- Suppose that the last four voters decided to rank B over C. Now who is the winner?

- The plurality-with-elimination Method may violate the Monotonicity Criterion.
- The other methods (I'm pretty sure) do not violate it.

Outline

The Independence-of-Irrelevant-Alternatives Criterion

Definition (Independence-of-Irrelevant-Alternatives Criterion)

The independence-of-irrelevant-alternatives criterion (IIA) says that if candidate X is the winner, then X would still be the winner if one or more of the losing candidates had not been in the race.

The Independence-of-Irrelevant-Alternatives Criterion

Definition (Independence-of-Irrelevant-Alternatives Criterion)

The independence-of-irrelevant-alternatives criterion (IIA) says that if candidate X is the winner, then X would still be the winner if one or more of the losing candidates had not been in the race.

- Check out the story of Sidney Morgenbesser.

Example (The Independence-of-Irrelevant-Alternatives Criterion – Plurality-with-Elimination Method)

	1	2	2
1st	A	A	B
2nd	B	B	A

- The group of 5 is offered a choice between Apple and Blueberry

The Independence-of-Irrelevant-Alternatives Criterion

> **Example (The Independence-of-Irrelevant-Alternatives Criterion – Plurality-with-Elimination Method)**
>
> | | 1 | 2 | 2 |
> |-----|---|---|---|
> | 1st | A | A | B |
> | 2nd | B | B | A |
>
> - The group of 5 is offered a choice between Apple and Blueberry
> - By "elimination" they choose Apple.

Example (The Independence-of-Irrelevant-Alternatives Criterion – Plurality-with-Elimination Method)

	1	2	2
1st	A	C	B
2nd	B	A	A
3rd	C	B	C

- The waitress comes back and includes Cherry as a third option.

Example (The Independence-of-Irrelevant-Alternatives Criterion – Plurality-with-Elimination Method)

	1	2	2
1st	A	C	B
2nd	B	A	A
3rd	C	B	C

- The waitress comes back and includes Cherry as a third option.
- Now which pie do they choose, by elimination?

Example (The Independence-of-Irrelevant-Alternatives Criterion – Plurality-with-Elimination Method)

	1	2	2
1st	A	C	B
2nd	B	A	A
3rd	C	B	C

- Independence of Irrelevant Alternatives runs that example in reverse.

The Independence-of-Irrelevant-Alternatives Criterion

Example (The Independence-of-Irrelevant-Alternatives Criterion – Plurality-with-Elimination Method)

	1	2	2
1st	A	C	B
2nd	B	A	A
3rd	C	B	C

- Independence of Irrelevant Alternatives runs that example in reverse.
- Who is the winner by the Plurality-with-Elimination Method?

The Independence-of-Irrelevant-Alternatives Criterion

Example (The Independence-of-Irrelevant-Alternatives Criterion – Plurality-with-Elimination Method)

	1	2	2
1st	A	C	B
2nd	B	A	A
3rd	C	B	C

- Independence of Irrelevant Alternatives runs that example in reverse.
- Who is the winner by the Plurality-with-Elimination Method?
- Suppose that candidate C drops out. Now who is the winner?

- The plurality method and the plurality-with-elimination method may violate the IIA Criterion.
- The other methods (I'm pretty sure) do not violate it.

Outline

Arrow's Impossibility Theorem

Theorem (Arrow's Impossibility Theorem)

If there are at least 3 candidates, then there is no voting method that cannot violate any of the four desired properties (Majority, Condorcet, Monotonicity, Independence of Irrelevant Alternatives).

Outline

Assignment

Assignment

- Chapter 1 Exercises 51, 52, 53, 54, 55, 56.